

DOUBLE POROSITY MODELS FOR ABSOLUTELY RIGID BODY VIA REITERATED HOMOGENIZATION

Anvarbek Meirmanov

Abstract

Double porosity models for the liquid filtration in an absolutely rigid body is derived from homogenization theory. The governing equations of the fluid dynamics on the microscopic level consist of the Stokes system for a slightly compressible viscous fluid, occupying a crack – pore space. In turn, this domain is a union of two independent systems of cracks (fissures) and pores. We suppose that the dimensionless size δ of pores depends on the dimensionless size ε of cracks: $\delta = \varepsilon^r$ with $r > 1$. The rigorous justification is fulfilled for homogenization procedure as the dimensionless size of the cracks tends to zero, while the solid body is geometrically periodic. As a result, for the long-time process we derive the usual Darcy equations of filtration for the liquid in cracks, while the liquid in pores is blocked and unmoved. For the short-time processes the homogenized system consists of acoustic equations, describing a two-velocity continuum with three independent parameters: the liquid velocity in pores, the liquid velocity in cracks and the common pressure. The proofs are based on the method of reiterated homogenization, suggested by G. Allaire and M. Briane.

Key words: Stokes equations; reiterated homogenization; Darcy law.

MOS subject classification: 35M99;76Q05

Introduction

The paper concerns a liquid motion in an absolutely rigid porous body. By the present moment there are a lot of different mathematical models, describing this physical process. They take into account a geometry of a space, occupied by the liquid (liquid domain), and physical properties of the liquid and the solid components. Among different models the simplest one is Darcy equations of filtration

$$\mathbf{v} = -k\nabla q + \mathbf{F}, \quad \nabla \cdot \mathbf{v} = 0, \quad (0.1)$$

for the macroscopic velocity \mathbf{v} and the pressure q of the liquid, when the solid skeleton is supposed to be an absolutely rigid body and the liquid domain is a pore space. For more complicate geometry, when the liquid domain is a union of system of pores and cracks, there are different phenomenological models (see, for example, Ref. [3], Ref. [8], Ref. [16], Ref. [17]). Note, that pores differ from cracks by its characteristic size: if l_p is a characteristic size of pores and l_c is a characteristic size of cracks, then $l_p \ll l_c$. The well-known phenomenological double-porosity model, suggested by G. I. Barenblatt, Iu. P. Zheltov and I. N. Kochina [3], describes two-velocity liquid continuum in absolutely rigid body, where macroscopic velocity \mathbf{v}_p and pressure q_p in pores and macroscopic velocity \mathbf{v}_c and pressure q_c in cracks satisfy two different Darcy laws

$$\mathbf{v}_p = -k_p \nabla q_p + \mathbf{F}, \quad \mathbf{v}_c = -k_c \nabla q_c + \mathbf{F}, \quad (0.2)$$

and two continuity equations

$$\nabla \cdot \mathbf{v}_p = J, \quad \nabla \cdot \mathbf{v}_c = -J. \quad (0.3)$$

The model is completed by postulate

$$J = \beta(q_c - q_p), \quad \beta = \text{const.}$$

Scientific and practical value of mathematical models describing such complicate processes, is obvious. But their physical reliability is also very important. Namely, we say, that the given phenomenological model *is physically correct* if it is one of basic models of continuum mechanics (as, for example, Stokes equations describing a slow motion of a viscous liquid, or Lamé's equations describing a motion of an elastic solid body) or asymptotically closed to some physically correct phenomenological model on the microscopic level (that is a model, obtained by homogenization of some model on the microscopic level, depending on the small parameter). In their fundamental paper R. Burridge and J. Keller [6] have used the very natural scheme to justify a physical correctness of the well – known in contemporary acoustics and filtration phenomenological model of poroelasticity, suggested by M. Biot [4]. As a model of the porous medium on the microscopic level authors have considered the mathematical model, consisting of Stokes equations describing liquid motion in pores and cracks and Lamé's equations, describing motion of a solid skeleton. The differential equations in the solid skeleton and in the liquid domain are completed by boundary conditions on the common boundary “liquid domain – solid skeleton”, which express a continuity of displacements and normal tensions. The suggested microscopic model is a basic one, because it follows from basic laws of continuum mechanics (see also E. Sanchez – Palencia [15]). After scaling there appears a natural small parameter δ which is the pore characteristic size l_p divided by the characteristic size L of the entire porous body: $\delta = l_p/L$. The small parameter enters both

into coefficients of the differential equations, and in the geometry of the domain in consideration. The homogenization (that is a finding of all limiting regimes as $\delta \searrow 0$) of this model is a model, asymptotically closed to the basic model and, by definition, is a physically correct. But even this approach is too difficult to be realized, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, it is most expedient to simplify the problem by postulating that the porous structure is periodic with the period δ . Under this assumption R. Burridge and J. Keller, using a method of two-scale asymptotic expansion, have formally proved a physical correctness of M. Biot's model. That is, the homogenized model derived by authors coincides with M. Biot's phenomenological model.

For the same geometry of the pore space (let call such a geometry as a *single porosity geometry* and corresponding mathematical model as a *single porosity model*) in absolutely rigid solid skeleton, when a liquid motion is described by the Stokes system

$$\alpha_\mu \Delta \mathbf{v} - \nabla q + \mathbf{F} = 0, \quad \nabla \cdot \mathbf{v} = 0,$$

for dimensionless microscopic velocity \mathbf{v} and dimensionless microscopic pressure q , where

$$\alpha_\mu = \mu_2 \delta^2, \quad \mu_1 = \text{const} > 0,$$

τ is the characteristic time of the process, ρ_f is the mean density of the liquid, g is the value of acceleration due to gravity, μ is the fluid viscosity, and $\mathbf{F}(\mathbf{x}, t)$ is the given dimensionless vector of distributed mass forces, the rigorous justification of the physical correctness of Darcy's law has been done by L. Tataru (see Appendix in Ref. [15]). Later a physical correctness of M. Biot's models, under the same assumptions on the geometry of a pore space as in Ref. [6], has been rigorously proved in Ref. [10] – Ref. [14].

For more complicate geometry of the absolutely rigid body, when the liquid domain is a crack – pore space (let call such a geometry as a *double porosity geometry* and corresponding mathematical model as a *double porosity model*), that is when a solid skeleton is perforated by system of pores and system of cracks, some attempts to derive macroscopic models, asymptotically closed to some phenomenological models on the microscopic level, have been made by T. Arbogast *et al* [2], A. Bourgeat *et al* [5] and Z. Chen [7]. Because the last two papers repeat ideas of the first one, let us briefly discuss the main idea in Ref. [2]. As an initial model on the microscopic level the authors have considered a periodic structure, consisting of “solid” blocks of the size ε surrounded by the fluid. The solid component is assumed to be already homogenized: there is no pore space and the motion of the fluid in blocks is governed by usual Darcy equations of filtration. The motion of the fluid in crack space (the space between “solid” blocks) is described by some artificial system, similar to Darcy equations of filtration. There is no any

physical base, but from mathematical point of view, such a choice of equations of fluid dynamics in cracks has a very solid base: it is impossible to find reasonable boundary conditions on the common boundary “solid” block-crack space, if the fluid dynamics is described by the Stokes equations. But there are reasonable boundary conditions, if the liquid motion is described by Darcy equations of filtration. Therefore, the final homogenized models in Ref. [2], Ref. [5] and in Ref. [7] are *physically incorrect*.

In the present publication we follow the scheme, suggested by R. Burridge and J. Keller [6], and look for double porosity models in absolutely rigid porous body, derived from the *basic model* on the microscopic level. In other words, we look for *physically correct macroscopic models*.

We start with a liquid domain, composed by a periodic system of pores with dimensionless size δ and a periodic system of cracks with dimensionless size ε , where $\delta = \varepsilon^r$. The liquid motion on the microscopic level is described by the Stokes system (see Ref.[9])

$$\alpha_\tau \frac{\partial \mathbf{v}}{\partial t} = \alpha_\mu \Delta \mathbf{v} - \nabla q + \mathbf{F}, \quad \frac{\partial q}{\partial t} + \alpha_q \nabla \cdot \mathbf{v} = 0, \quad (0.4)$$

for dimensionless microscopic velocity \mathbf{v} and pressure q of the liquid, where

$$\alpha_\mu = \frac{2\mu}{\tau L g \rho_f}, \quad \alpha_\tau = \frac{L}{g \tau^2}, \quad \alpha_q = \frac{c^2}{L g},$$

and c is a speed of sound in fluid.

The case $r = 1$ corresponds to already studied situation of a simple pore space, and the case $r > 1$ corresponds to a real double porosity geometry.

We assume that all dimensionless parameters depend on the small parameter ε and the (finite or infinite) limits exist:

$$\lim_{\varepsilon \searrow 0} \alpha_\tau(\varepsilon) = \tau_0, \quad \lim_{\varepsilon \searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon \searrow 0} \alpha_q(\varepsilon) = c_f^2, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\delta^2} = \mu_2.$$

In what follows we deal with cases, when

$$\mu_0 = 0, \quad \tau_0 < \infty, \quad 0 < c_f < \infty.$$

The aim of any homogenization procedure is to find all possible limiting regimes as $\varepsilon \searrow 0$. Of course, these regimes for the model (0.4) depend on criteria τ_0 and μ_1 , which characterize different type of physical processes. We may roughly divide all these processes into two groups: long-time processes (filtration) and short-time processes (acoustics). It is well-known, that the characteristic time of the liquid filtration is about month, while the characteristic size of the entire porous body is about thousand meters. Therefore, we may assume that for filtration $\tau_0 = 0$. The rest of processes we call acoustics and all these situations characterized by criterion $\tau_0 > 0$.

If we consider the case of simple geometry ($r = 1$) in filtration, then the homogenization procedure has a sense only if $\mu_2 > 0$ (see Ref. [10]). Moreover, if $\mu_2 = \infty$ (extremely viscous liquid), then the unique limiting regime is an immobility (the liquid is blocked in pores). This fact is a simple consequence of the Friedrichs-Poincaré inequality. The same situation is repeated for the case $r > 1$ of double porosity geometry. As before, the homogenization procedure has a sense if and only if $\mu_1 > 0$. But this criterion automatically implies the equality $\mu_2 = \infty$. Therefore, due to the same Friedrichs-Poincaré inequality the limiting regime for the liquid in pores is a rest state (that is the velocity \mathbf{v}_p in pores is identically zero). Next we use the method of reiterated homogenization, suggested by G. Allaire and M. Briane[1] and for the velocity of the liquid in cracks \mathbf{v}_c and liquid pressure q , when the crack space is connected, we derive the usual Darcy equations of filtration:

$$\mathbf{v}_c = \mathbb{B}_c^{(1)} \cdot (\mathbf{F} - \frac{1}{m} \nabla q), \quad \frac{1}{c_f^2} \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{v}_c = 0, \quad \mathbf{v}_p \equiv 0. \quad (0.5)$$

For disconnected crack space (isolated cracks) the unique limiting regime is a rest state: $\mathbf{v}_c \equiv 0$.

*Note, that under condition $\mu_2 = \infty$ the limiting regime for the liquid in pores in the single porosity model is also a rest state, as in the double – porosity model. **Therefore, the main point here is not in the geometry of the solid skeleton, but only in physical assumptions.***

It is absolutely clear, that there is an overflow from pores to cracks and vice versa for the double porosity geometry, and there is a very slow flow in pores for the single porosity geometry in the real physical processes. Therefore, the initial postulates of a basic model (more precisely, the supposition that a solid skeleton is an absolutely rigid body) do not correspond to the reality. But, on the other hand, we can say that the obtained physically correct model describes the physical problem with a simulation error $o(c_s^{-1})$, where c_s is one of the speeds of sound in the solid skeleton.

Anyway, the double porosity models, suggested in Ref. [3], Ref. [2], Ref. [5], or in Ref. [7] have no common with a physically correct mathematical models. First of all, the correct models contain only one pressure, common both for pores and for cracks. Secondly, the liquid in pores is blocked and unmoved.

For short-time processes ($\tau_0 > 0$) the situation is different. Under this restriction the homogenization procedure has a sense for any μ_1 and μ_2 . In particular, for $\mu_1 = 0$ and $\mu_2 = 0$ we show that the limiting regime is a two-velocity continuum with three independent characteristics: velocity in pores \mathbf{v}_p , velocity in cracks \mathbf{v}_c and common pressure q , which satisfy the acoustic equations, consisting of two momentum conservation laws in the form

$$\tau_0 \frac{\partial \mathbf{v}_p}{\partial t} = (m_p \beta_c \mathbb{I} - \mathbb{B}_p^{(2)}) \cdot (-\frac{1}{m} \nabla q + \mathbf{F}), \quad (0.6)$$

$$\tau_0 \frac{\partial \mathbf{v}_c}{\partial t} = (m_c \mathbb{I} - \mathbb{B}_c^{(2)}) \cdot \left(-\frac{1}{m} \nabla q + \mathbf{F} \right), \quad (0.7)$$

and continuity equation

$$\frac{1}{c_f^2} \frac{\partial q}{\partial t} + \nabla \cdot (\mathbf{v}_c + \mathbf{v}_p) = 0. \quad (0.8)$$

For sufficiently large c_f (incompressible liquid) we may substitute (0.8) by

$$\nabla \cdot (\mathbf{v}_c + \mathbf{v}_p) = 0,$$

which coincides with the sum of continuity equations in (0.3), but equations (0.6) and (0.7) have completely different structure than equations (0.2), even for the case $q_p = q_c$.

§1. Mathematical model on a microscopic level and basic a' priori estimates

First of all we define the liquid domain Ω^ε , which is a subdomain of the unit cube Ω . Let $\Omega = Z_f \cup Z_s \cup \gamma_c$, where Z_f and Z_s are open sets, the common boundary $\gamma_c = \partial Z_f \cap \partial Z_s$ is a Lipschitz continuous surface, and a periodic repetition of the domain Z_s is a connected domain with a Lipschitz continuous boundary. The elementary cell Z_f models a crack space Ω_c^ε : the domain Ω_c^ε is an intersection of the cube Ω with a periodic repetition in \mathbb{R}^3 of the elementary cell εZ_f . In the same way we define the pore space Ω_p^δ : $\Omega = Y_f \cup Y_s \cup \gamma_p$, γ_c is a Lipschitz continuous surface, a periodic repetition of the domain Y_s is a connected domain with a Lipschitz continuous boundary, and Ω_p^δ is an intersection of $\Omega \setminus \Omega_c^\varepsilon$ with a periodic repetition in \mathbb{R}^3 of the elementary cell δY_f . Finally, we put $\Omega^\varepsilon = \Omega_p^\delta \cup \Omega_c^\varepsilon$.

We may also characterize a liquid domain using indicator functions. In fact, let $\eta(\mathbf{x})$ be the indicator function of the domain Ω in \mathbb{R}^3 , that is $\eta(\mathbf{x}) = 1$ if $\mathbf{x} \in \Omega$ and $\eta(\mathbf{x}) = 0$ if $\mathbf{x} \in \mathbb{R}^3 \setminus \Omega$. Let also $\chi_p(\mathbf{y})$ be the 1-periodic extension of the indicator function of the domain Y_f in Y and $\chi_c(\mathbf{z})$ be the 1-periodic extension of the indicator function of the domain Z_f in Z . Then $\chi_c^\varepsilon(\mathbf{x}) = \eta(\mathbf{x})\chi_c(\mathbf{x}/\varepsilon)$ stands for the indicator function of the domain Ω_c^ε , $\chi_p^\varepsilon(\mathbf{x}) = \eta(\mathbf{x})(1 - \chi_c(\mathbf{x}/\varepsilon))\chi_p(\mathbf{x}/\delta)$ stands for the indicator function of the domain Ω_p^δ and $\chi^\varepsilon(\mathbf{x}) = \chi_c^\varepsilon(\mathbf{x}) + \chi_p^\varepsilon(\mathbf{x})$ stands for the indicator function of the liquid domain Ω^ε .

Next we define the generalized solution of the problem (0.4) as functions \mathbf{v}^ε and q^ε such that

$$\mathbf{v}^\varepsilon \in L^2((0, T); \overset{\circ}{W}_2^1(\Omega^\varepsilon)) \quad \text{and} \quad \frac{\partial q^\varepsilon}{\partial t} \in L^2(\Omega^\varepsilon \times (0, T))$$

satisfying continuity equation in (0.4) in a usual sense almost everywhere in $\Omega^\varepsilon \times (0, T)$ and integral identity

$$\int_0^T \int_{\Omega^\varepsilon} \left(\alpha_\tau \mathbf{v}^\varepsilon \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} - \alpha_\mu \nabla \mathbf{v}^\varepsilon : \nabla \boldsymbol{\varphi} + q^\varepsilon \nabla \cdot \boldsymbol{\varphi} + \mathbf{F} \cdot \boldsymbol{\varphi} \right) dx dt = 0 \quad (1.1)$$

for any smooth vector-functions $\boldsymbol{\varphi}$, vanishing at $\partial\Omega^\varepsilon$ and $t = T$. The homogeneous boundary condition $\mathbf{v} = 0$ on $\partial\Omega^\varepsilon$ is already included into the functional space $W_2^1(\Omega^\varepsilon)$ and homogeneous initial condition $\mathbf{v}(\mathbf{x}, 0) = 0$ is included into integral identity (1.1). The known function \mathbf{F} is supposed L^2 integrable over domain $\Omega \times (0, T)$. In the same standard way, as in Ref. [10], one can show that for any $\varepsilon > 0$ there exists a unique generalized solution to the problem (0.4) and

$$\int_0^T \int_{\Omega^\varepsilon} \left(|\mathbf{v}^\varepsilon|^2 + \alpha_\mu |\nabla \mathbf{v}^\varepsilon|^2 + |q^\varepsilon|^2 + |\nabla \cdot \mathbf{v}^\varepsilon|^2 \right) dx dt \leq C \int_0^T \int_{\Omega^\varepsilon} |\mathbf{F}|^2 dx dt, \quad (1.2)$$

$$\int_0^T \left(\frac{\alpha_\mu}{\delta^2} \int_{\Omega} \chi_p^\varepsilon |\mathbf{v}^\varepsilon|^2 dx + \frac{\alpha_\mu}{\varepsilon^2} \int_{\Omega} \chi_c^\varepsilon |\mathbf{v}^\varepsilon|^2 dx \right) dt \leq C \int_0^T \int_{\Omega^\varepsilon} |\mathbf{F}|^2 dx dt, \quad (1.3)$$

where C is independent of ε .

To prove (1.2) and (1.3) we multiply the Stokes equation in (0.4) for \mathbf{v}^ε by \mathbf{v}^ε and integrate by parts over domain Ω^ε

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega^\varepsilon} \left(\alpha_\tau |\mathbf{v}^\varepsilon|^2 + \frac{1}{\alpha_q} |q^\varepsilon|^2 \right) dx + \int_{\Omega^\varepsilon} \alpha_\mu |\nabla \mathbf{v}^\varepsilon|^2 dx = \int_{\Omega^\varepsilon} \mathbf{F} \cdot \mathbf{v}^\varepsilon dx. \quad (1.4)$$

For $\tau_0 > 0$ (1.2) is a simple consequence of (1.4). Let now $\tau_0 = 0$. To estimate $I^\varepsilon = \int_{\Omega^\varepsilon} |\mathbf{v}^\varepsilon|^2 dx$ we divide it by two integrals:

$$I^\varepsilon = I_p^\delta + I_c^\varepsilon, \quad I_p^\delta = \int_{\Omega_p^\delta} |\mathbf{v}^\varepsilon|^2 dx, \quad I_c^\varepsilon = \int_{\Omega_c^\varepsilon} |\mathbf{v}^\varepsilon|^2 dx.$$

Let $G_p^{(\mathbf{k})}$, where $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3$, be the intersection of Ω_p^δ with a set $\{\mathbf{x} : \mathbf{x} = \varepsilon(\mathbf{y} + \mathbf{k}), \mathbf{y} \in Y\}$. Then $\Omega_p^\delta = \cup_{\mathbf{k} \in \mathbb{Z}^3} G_p^{(\mathbf{k})}$ and

$$I_p^\delta = \sum_{\mathbf{k} \in \mathbb{Z}^3} I_p^\delta(\mathbf{k}), \quad I_p^\delta(\mathbf{k}) = \int_{G_p^{(\mathbf{k})}} |\mathbf{v}^\varepsilon|^2 dx.$$

In each integral I_p^δ we change variable by $\mathbf{x} = \varepsilon \mathbf{y}$, next apply the Friedrichs-Poincaré inequality and finally return to original variables:

$$\int_{G_p^{(\mathbf{k})}} |\mathbf{v}^\varepsilon|^2 dx = \delta^3 \int_{Y^{(\mathbf{k})}} |\bar{\mathbf{v}}^\varepsilon|^2 dy \leq \delta^3 C^{(\mathbf{k})} \int_{Y^{(\mathbf{k})}} |\nabla_y \bar{\mathbf{v}}^\varepsilon|^2 dy = \delta^2 C^{(\mathbf{k})} \int_{G_p^{(\mathbf{k})}} |\nabla_x \mathbf{v}^\varepsilon|^2 dx.$$

Here $\bar{\mathbf{v}}^\varepsilon(\mathbf{y}, t) = \mathbf{v}^\varepsilon(\mathbf{x}, t)$, $Y^{(\mathbf{k})} \subset Y$ is an appropriate translation to origin of the set $(1/\delta)G_p^{(\mathbf{k})}$, and $C^{(\mathbf{k})}$ is a constant in the Friedrichs-Poincaré inequality for the domain $Y^{(\mathbf{k})}$. To estimate these constants uniformly with respect to δ (or ε) let us clarify the structure of the domain $Y^{(\mathbf{k})}$. If the closure of $G_p^{(\mathbf{k})}$ has no intersection with the boundary between pore and crack spaces, then $Y^{(\mathbf{k})} = Y_f$ and $C^{(\mathbf{k})}$ coincides with a fixed constant C . Otherwise, $Y^{(\mathbf{k})}$ is one of two domains, obtained after splitting Y_f by some smooth surface, asymptotically closed to the plane as $\varepsilon \searrow 0$. Due to supposition on the structure of the solid part Y_f , constants $C^{(\mathbf{k})}$ uniformly bounded for all possible planes, splitting Y_f . Therefore, $\sup C^{(\mathbf{k})} \leq C$ (for simplicity we denote all constants independent of ε as C) and

$$I_p^\delta \leq \delta^2 C \sum_{\mathbf{k} \in \mathbb{Z}^3} \int_{G_p^{(\mathbf{k})}} |\nabla_x \mathbf{v}^\varepsilon|^2 dx \leq \delta^2 C \int_{\Omega^\varepsilon} |\nabla_x \mathbf{v}^\varepsilon|^2 dx. \quad (1.5)$$

To explain ideas we consider the easiest geometry, when the liquid part Y_f is “surrounded” by the solid part Y_s . That is, for each facet $S \subset \partial Y$ of Y the liquid part $S \cap \partial Y_f$ is completely surrounded by the solid part $S \cap \partial Y_s$. Next we extend the velocity \mathbf{v}^ε from the liquid to the solid as zero. Then the constant in the Friedrichs-Poincaré inequality for $Y^{(\mathbf{k})}$ depend only on the ratio $\sigma = V_f/V_s$ between the volume V_f of the liquid part of $Y^{(\mathbf{k})}$ and the volume V_s of the solid part of $Y^{(\mathbf{k})}$: $C^{(\mathbf{k})} \leq C\sigma$. It is easy to see, that for chosen geometry of Y_f and for any type of splitting of Y by planes, this ratio σ is uniformly bounded.

In the same way we show that

$$I_c^\varepsilon \leq \varepsilon^2 C \int_{\Omega^\varepsilon} |\nabla_x \mathbf{v}^\varepsilon|^2 dx. \quad (1.6)$$

Thus,

$$I^\varepsilon \leq C \left(\frac{\delta^2}{\alpha_\mu} + \frac{\varepsilon^2}{\alpha_\mu} \right) \alpha_\mu \int_{\Omega^\varepsilon} |\nabla_x \mathbf{v}^\varepsilon|^2 dx.$$

The rest of the proof is standard and follows from (1.4) – (1.6) and Hölder and Gronwall’s inequalities.

§2. Homogenization procedure

Now we ready to find limiting regimes for the model (1.1). As the first step we extend the velocity and pressure from Ω^ε to Ω as zero, keeping for simplicity the same notations. Next we use estimate (1.2) and conclude, that under condition

$$\tau_0 + \mu_1 > 0 \quad (2.1)$$

(see Ref. [10]) the sequences $\{\mathbf{v}^\varepsilon\}$, $\{q^\varepsilon\}$ and $\{\nabla \cdot \mathbf{v}^\varepsilon\}$ weakly converge in $L^2(\Omega \times (0, T))$ (up to some subsequences) to \mathbf{v} , q and $\nabla \cdot \mathbf{v}$ correspondingly and

$$\frac{1}{c_f^2} \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{v} = 0. \quad (2.2)$$

At the same time sequences $\{\mathbf{v}^\varepsilon\}$ and $\{q^\varepsilon\}$ three – scale converge (up to some subsequences) to 1-periodic in variables \mathbf{y} and \mathbf{z} functions $\mathbf{V}(\mathbf{x}, t, \mathbf{y}, \mathbf{z})$ and $Q(\mathbf{x}, t, \mathbf{y}, \mathbf{z})$ correspondingly (see Ref. [1]) and

$$\int_Y \int_Z \mathbf{V} dz dy = \mathbf{v}, \quad \int_Y \int_Z Q dz dy = q.$$

Note, that *three – scale convergence* means the convergence of integrals

$$\int_0^T \int_\Omega \mathbf{v}^\varepsilon(\mathbf{x}, t) \cdot \boldsymbol{\varphi}(\mathbf{x}, t, \frac{\mathbf{x}}{\varepsilon}, \frac{\mathbf{x}}{\delta}) dx dt \rightarrow \int_0^T \int_\Omega \int_Y \int_Z \mathbf{V}(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) \cdot \boldsymbol{\varphi}(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) dz dy dx dt,$$

for any smooth 1-periodic in \mathbf{y} and \mathbf{z} function $\boldsymbol{\varphi}(\mathbf{x}, t, \mathbf{y}, \mathbf{z})$.

If we define a microscopic velocity in pores \mathbf{v}_p^ε as $\mathbf{v}_p^\varepsilon = \chi_p^\varepsilon \mathbf{v}^\varepsilon$ and a microscopic velocity in cracks \mathbf{v}_c^ε as $\mathbf{v}_c^\varepsilon = \chi_c^\varepsilon \mathbf{v}^\varepsilon$, then it is naturally to call a weak limit \mathbf{v}_p of the sequence $\{\mathbf{v}_p^\varepsilon\}$ as a macroscopic velocity in pores and a weak limit \mathbf{v}_c of the sequence $\{\mathbf{v}_c^\varepsilon\}$ as a macroscopic velocity in cracks. On the other hand, by definition of three – scale convergence, the sequences $\{\mathbf{v}_p^\varepsilon\}$ and $\{\mathbf{v}_c^\varepsilon\}$ three – scale converge to $(1 - \chi_c(\mathbf{z}))\chi_p(\mathbf{y})\mathbf{V}(\mathbf{x}, t, \mathbf{y}, \mathbf{z})$ and $\chi_c(\mathbf{z})\mathbf{V}(\mathbf{x}, t, \mathbf{y}, \mathbf{z})$ correspondingly and

$$\int_Y \int_Z (1 - \chi_c)\chi_p \mathbf{V} dz dy = \mathbf{v}_p, \quad \int_Y \int_Z \chi_c \mathbf{V} dz dy = \mathbf{v}_c, \quad \mathbf{v}_p + \mathbf{v}_c = \mathbf{v}. \quad (2.3)$$

As the next step we state that $Q = q(\mathbf{x}, t)\chi(\mathbf{y}, \mathbf{z})/m$, where $\chi = \chi_c(\mathbf{z}) + (1 - \chi_c(\mathbf{z}))\chi_p(\mathbf{y})$, $m = m_c + (1 - m_c)m_p$ is the porosity of the crack – pore space, $m_c = \int_Z \chi_c dz$ is the porosity of the crack space and $m_p = \int_Y \chi_p dy$ is the porosity of the pore space. The proof of this fact repeats the corresponding proof in Ref. [10].

To find function \mathbf{V} we must derive some microscopic periodic boundary-value problems in the domain $W = \{(\mathbf{y}, \mathbf{z}) : \chi(\mathbf{y}, \mathbf{z}) = 1\}$. It is clear, that these problems somehow follow from the basic integral identity (1.1) as $\varepsilon \searrow 0$. The condition $\text{supp } \boldsymbol{\varphi} \subset \Omega^\varepsilon$ for the test function $\boldsymbol{\varphi}$ in (1.1) means that $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{x}, t, \mathbf{x}/\varepsilon, \mathbf{x}/\delta)$. This form of the test function creates difficulties in the limiting procedure. In following terms

$$J_1^\varepsilon(t) \equiv \int_\Omega q^\varepsilon \nabla \cdot \boldsymbol{\varphi} dx = \int_\Omega q^\varepsilon (\nabla_x \cdot \boldsymbol{\varphi} + \frac{1}{\varepsilon} \nabla_z \cdot \boldsymbol{\varphi} + \frac{1}{\delta} \nabla_y \cdot \boldsymbol{\varphi}) dx,$$

$$J_2^\varepsilon(t) \equiv \int_\Omega \alpha_\mu \nabla \mathbf{v}^\varepsilon : \nabla \boldsymbol{\varphi} dx = - \int_\Omega \alpha_\mu \mathbf{v}^\varepsilon \cdot \left(\Delta_x \boldsymbol{\varphi} + \frac{2}{\varepsilon} \nabla_x \cdot (\nabla_z \boldsymbol{\varphi}) + \right.$$

$$\frac{2}{\delta} \nabla_x \cdot (\nabla_y \varphi) + \frac{2}{\varepsilon \delta} \nabla_z \cdot (\nabla_y \varphi) + \frac{1}{\varepsilon^2} \Delta_z \varphi + \frac{1}{\delta^2} \Delta_y \varphi \Big) dx.$$

in the integral identity (1.1) appear unbounded items as $\varepsilon \searrow 0$.

The simple analysis of (1.3) shows that for nontrivial homogenization procedure $\mu_1 < \infty$. Otherwise, $\mathbf{v}_p^\varepsilon \rightarrow 0$, $\mathbf{v}_c^\varepsilon \rightarrow 0$ as $\varepsilon \searrow 0$. Next we separate different cases.

2.1. Liquid filtration

For filtration processes $\tau_0 = 0$ and $\mu_1 > 0$. Therefore, $\mu_2 = \infty$ and (1.3) implies $\mathbf{v}_p = 0$.

As a test function in (1.1) we choose the function $\varphi = h(\mathbf{x}, t) \psi(\mathbf{x}/\varepsilon)$, where $h(\mathbf{x}, t)$ vanishes at $\partial\Omega$ and at $t = T$ and function $\psi(\mathbf{z})$ is finite in Z_f ($\text{supp } \psi \subset Z_f$) and solenoidal ($\nabla_z \cdot \psi = 0$).

Due to relations

$$\lim_{\varepsilon \searrow 0} \alpha_\tau(\varepsilon) = 0, \quad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\varepsilon} = 0, \quad \nabla_y \varphi = 0, \quad \Delta_y \varphi = 0, \quad \nabla_y \cdot \varphi = \nabla_z \cdot \varphi = 0,$$

a three - scale limit in (1.1) yields integral identity

$$\int_0^T \int_\Omega \int_{Z_f} \left((\mu_1 \mathbf{V}_c \cdot \Delta_z \psi) h + (h \mathbf{F} + \frac{q}{m} \nabla h) \cdot \psi \right) dz dx dt = 0,$$

for the function

$$\mathbf{V}_c = \int_Y \mathbf{V}(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) dy.$$

Reintegrating this identity with respect to variables (\mathbf{x}, t) we arrive at

$$0 = \int_{Z_f} \left(\mu_1 \mathbf{V}_c \cdot \Delta_z \psi + (\mathbf{F} - \frac{1}{m} \nabla q) \cdot \psi \right) dz = \int_{Z_f} (\mu_1 \Delta_z \mathbf{V}_c + \mathbf{F} - \frac{1}{m} \nabla q) \cdot \psi dz,$$

and consequently

$$\mu_1 \Delta_z \mathbf{V}_c(\mathbf{x}, t, \mathbf{z}) - \nabla_z \Pi_c(\mathbf{x}, t, \mathbf{z}) + \mathbf{F}(\mathbf{x}, t) - \frac{1}{m} \nabla q(\mathbf{x}, t) = 0, \quad \mathbf{z} \in Z_f. \quad (2.4)$$

The term $\nabla_z \Pi_c$ appears due to condition $\nabla_z \cdot \psi = 0$.

Three-scale limit in the continuity equation in (0.4) in the form

$$\int_0^T \int_\Omega \left(q^\varepsilon \frac{\partial \psi}{\partial t} - \alpha_q \nabla \psi \cdot \mathbf{v}^\varepsilon \right) dx dt = 0 \quad (2.5)$$

with test function $\psi = \varepsilon h(\mathbf{x}, t) \psi_0(\mathbf{x}/\varepsilon)$ yields

$$\nabla_z \cdot \mathbf{V}_c = 0, \quad \mathbf{z} \in Z_f. \quad (2.6)$$

Finally, the representation $\mathbf{V}_c = \chi_c(\mathbf{z}) \mathbf{V}_c$ and supposition $\mu_1 > 0$ (for this case the function $\nabla_z(\mathbf{V}_c)$ is L^2 -integrable in Z_f) give us the boundary condition

$$\mathbf{V}_c = 0, \quad \mathbf{z} \in \gamma_c. \quad (2.7)$$

The problem (2.4), (2.6) – (2.7) is well-known (see Ref. [15] and Ref. [10]) and has a unique solution in the form

$$\mathbf{V}_c = \mathbb{A}_c^{(1)}(\mathbf{z}) \cdot (\mathbf{F} - \frac{1}{m} \nabla q),$$

or

$$\mathbf{v}_c = \mathbb{B}_c^{(1)} \cdot (\mathbf{F} - \frac{1}{m} \nabla q), \quad \mathbb{B}_c^{(1)} = \int_{Z_f} \mathbb{A}_c^{(1)}(\mathbf{z}) d\mathbf{z}, \quad (2.8)$$

which is exactly the Darcy law for the liquid in the cracks. Equation (2.8) together with continuity equation (2.2) and relations (2.3), where $\mathbf{v}_p = 0$, constitute the homogenized double-porosity model:

$$\mathbf{v}_c = \mathbb{B}_c^{(1)} \cdot (\mathbf{F} - \frac{1}{m} \nabla q), \quad \frac{1}{c_f^2} \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{v}_c = 0, \quad \mathbf{v}_p \equiv 0. \quad (2.9)$$

For disconnected crack space (we define a *disconnected crack space* as a structure where the liquid part Z_f is completely surrounded by the solid part Z_s : $\partial Z \cap \gamma_c = \emptyset$) the problem (2.4), (2.6) – (2.7) has a unique solution $\mathbf{V}_c = 0$. That is *for disconnected crack space the liquid in cracks is also unmoved*.

2.2. Acoustics

For acoustics $\tau_0 > 0$. As we have mentioned above, under this condition the homogenization procedure has a sense for any $\mu_1 < \infty$. To explain ideas, we consider only the case $\mu_1 = \mu_2 = 0$. Under these restrictions the term $I_2^\varepsilon(t)$ in (1.1) goes to zero as $\varepsilon \searrow 0$. It is clear, that we may pass to limit in $I_1^\varepsilon(t)$ only for test functions φ satisfying

$$\nabla_z \cdot \varphi = \nabla_y \cdot \varphi = 0.$$

We have already used one of the possible ways to find microscopic equations (2.4). To derive microscopic equations for pore space we need a test function φ depending on both variable \mathbf{y} and \mathbf{z} . Such a choice is possible, if we put

$$\varphi = h(\mathbf{x}, t) \left(\nabla_z \varphi_c \left(\frac{\mathbf{x}}{\varepsilon} \right) \times \nabla_y \varphi_p \left(\frac{\mathbf{x}}{\delta} \right) \right), \quad \text{supp } \varphi_c(\mathbf{z}) \subset Z_s, \quad \text{supp } \varphi_p(\mathbf{y}) \subset Y_f.$$

In fact, using the easy checking formula

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot \text{rot } \mathbf{b} - \mathbf{b} \cdot \text{rot } \mathbf{a}$$

we have

$$\begin{aligned} \nabla_z \cdot \varphi &= -h(\nabla_y \varphi_p(\mathbf{y})) \cdot \left(\text{rot}_z (\nabla_z \varphi_c(\mathbf{z})) \right) = 0, \\ \nabla_y \cdot \varphi &= h(\nabla_z \varphi_c(\mathbf{z})) \cdot \left(\text{rot}_y (\nabla_y \varphi_p(\mathbf{y})) \right) = 0. \end{aligned}$$

Therefore

$$J_1^\varepsilon(t) = \int_{\Omega} q^\varepsilon \nabla_x h \cdot \left(\nabla_z \varphi_c \left(\frac{\mathbf{x}}{\varepsilon} \right) \times \nabla_y \varphi_p \left(\frac{\mathbf{x}}{\delta} \right) \right) dx$$

and the limit as $\varepsilon \searrow 0$ in (1.1) yields

$$\int_0^T \int_{\Omega} \int_Y \int_Z \left(\tau_0 \mathbf{V} \frac{\partial h}{\partial t} + \frac{1}{m} q \nabla_x h + h \mathbf{F} \right) \cdot \left(\nabla_z \varphi_c(\mathbf{z}) \times \nabla_y \varphi_p(\mathbf{y}) \right) dy dz dx dt = 0.$$

First, we reintegrate this integral identity with respect to variables (\mathbf{x}, t)

$$\int_Y \int_Z \mathbf{U} \cdot (\nabla_z \varphi_c(\mathbf{z}) \times \nabla_y \varphi_p(\mathbf{y})) dy dz = 0, \quad \mathbf{U} = \tau_0 \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{m} \nabla_x q - \mathbf{F},$$

and after that, reintegrate the last identity with respect to variables (\mathbf{y}, \mathbf{z}) :

$$\nabla_y \cdot (\text{rot}_z \mathbf{U}) = \nabla_z \cdot (\text{rot}_y \mathbf{U}) = 0.$$

This equation has a solution

$$\mathbf{U} = \nabla_y Q_p(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) + \nabla_z Q_c(\mathbf{x}, t, \mathbf{y}, \mathbf{z}),$$

1-periodic in variables (\mathbf{y}, \mathbf{z}) , which leads to the desired microscopic equation

$$\tau_0 \frac{\partial \mathbf{V}}{\partial t} = -\nabla_y Q_p - \nabla_z Q_c - \frac{1}{m} \nabla_x q + \mathbf{F} \quad (2.10)$$

in the domain $W \subset Y \otimes Z$. Equation (2.10) is completed with continuity equation (2.6) and continuity equation

$$\nabla_y \cdot \mathbf{V} = 0, \quad (\mathbf{y}, \mathbf{z}) \in W, \quad (2.11)$$

which is a result of three – scale limit in (0.4) for test functions in the form

$$\psi = \delta h(\mathbf{x}, t) \psi_0\left(\frac{\mathbf{x}}{\varepsilon}\right) \psi_1\left(\frac{\mathbf{x}}{\delta}\right).$$

Now we derive the microscopic equations in the crack space for functions

$$\mathbf{V}_c(\mathbf{x}, t, \mathbf{z}) = \int_Y \mathbf{V}(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) dy, \quad \Pi_c(\mathbf{x}, t, \mathbf{z}) = \int_Y Q_c(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) dy.$$

To do that we just integrate (2.10) over Y and take into account the periodicity of the function Q_p :

$$\tau_0 \frac{\partial \mathbf{V}_c}{\partial t} = -\nabla_z \Pi_c - \frac{1}{m} \nabla_x q + \mathbf{F}, \quad \mathbf{z} \in Z_f. \quad (2.12)$$

Equations (2.6), (2.12) are endowed with homogeneous initial and boundary conditions

$$\mathbf{V}_c(\mathbf{x}, 0, \mathbf{z}) = 0, \quad \mathbf{z} \in Z_f, \quad \mathbf{V}_c(\mathbf{x}, t, \mathbf{z}) \cdot \mathbf{n}(\mathbf{z}) = 0, \quad \mathbf{z} \in \gamma_c, \quad (2.13)$$

where $\mathbf{n}(\mathbf{z})$ is the unit normal vector to γ_c at a point $\mathbf{z} \in \gamma_c$. The boundary condition in (2.13) is a consequence of the representation $\mathbf{V}_c = \chi_c(\mathbf{z})\mathbf{V}_c$ and continuity equation (2.6).

The problem (2.6), (2.12), (2.13) is well-known (see Ref. [15] and Ref. [10]) and has a unique solution in the form

$$\nabla_z \Pi_c = \mathbb{A}_c^{(2)}(\mathbf{z}) \cdot \left(-\frac{1}{m} \nabla_x q + \mathbf{F}\right), \quad \mathbb{B}_c^{(2)} = \int_{Z_f} \mathbb{A}_c^{(2)}(\mathbf{z}) dz. \quad (2.14)$$

Finally, to derive macroscopic equation (0.6) we just integrate (2.12) over domain Z_f and take into account (2.14).

As a last step we must find some microscopic equations for two-scale microscopic velocity and pressure

$$\mathbf{V}_p(\mathbf{x}, t, \mathbf{y}) = \int_{Z_s} \mathbf{V}(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) dz, \quad \Pi_p(\mathbf{x}, t, \mathbf{y}) = \int_{Z_s} Q_p(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) dz$$

in pores, when $\mathbf{y} \in Y_f$. The main problem here is a definition of the term

$$\mathbf{G}(\mathbf{x}, t, \mathbf{y}) = \int_{\gamma_c} Q_c(\mathbf{x}, t, \mathbf{y}, \mathbf{z}) \mathbf{n}(\mathbf{z}) d\sigma_z,$$

which appears after integration (2.10) over domain Z_s :

$$\tau_0 \frac{\partial \mathbf{V}_p}{\partial t} = -\nabla_y \Pi_p - \mathbf{G} + (1 - m_c) \left(-\frac{1}{m} \nabla_x q + \mathbf{F}\right), \quad \mathbf{y} \in Y_f. \quad (2.15)$$

This function \mathbf{G} stands for the interaction between pore and crack spaces. There is no any additional equation, which may define \mathbf{G} , and to find at least the structure of homogenized equations for short-time processes in the model (0.4) we postulate the relation

$$\mathbf{G} = -\beta \left(-\frac{1}{m} \nabla_x q + \mathbf{F}\right), \quad \beta = \text{const} > 0. \quad (2.16)$$

Under restriction (2.16) equation (2.15), continuity equation

$$\nabla_y \cdot \mathbf{V}_p = 0, \quad \mathbf{y} \in Y_f, \quad (2.17)$$

and initial and boundary conditions

$$\mathbf{V}_p(\mathbf{x}, 0, \mathbf{y}) = 0, \quad \mathbf{y} \in Y_f, \quad \mathbf{V}_p(\mathbf{x}, t, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) = 0, \quad \mathbf{y} \in \gamma_p, \quad (2.18)$$

uniquely define the homogenized equation

$$\tau_0 \frac{\partial \mathbf{v}_p}{\partial t} = (m_p \beta_c \mathbb{I} - \mathbb{B}_p^{(2)}) \cdot \left(-\frac{1}{m} \nabla q + \mathbf{F}\right), \quad (2.19)$$

where $\beta_c = \beta + 1 - m_c > 0$ (see representation (2.14)).

Equations (0.6) – (0.8) constitute the closed homogenized system, describing acoustics in absolutely rigid body perforated by systems of pores and cracks, filled by slightly compressible liquid.

Conclusions

We have shown how the new rigorous homogenization method, suggested by G. Allaire and M. Briane, can be used to clarify the structure of mathematical models for liquid filtration in natural reservoirs with very complicate geometry. Using the scheme, suggested by R. Burridge and J. Keller, we have shown that for **double porosity geometry** of the **absolutely rigid solid skeleton**

1) the unique physically correct model for liquid filtration is a usual Darcy system for the liquid in cracks while the liquid in pores is blocked,

and

2) acoustics described by some physically correct model for two independent velocities in pores and cracks and for common pressure.

Acknowledgment

This research is partially supported by Russian Foundation of Basic Research under grant number 08-05-00265.

References

- [1] G. Allaire and M. Briane, Multiscale convergence and reiterated homogenization, *Proceed. of Royal Soc. Edinburgh* **126A** (1996) 297-342.
- [2] T. Arbogast, J. Jn. Douglas and U. Hornung, Derivation of the double-porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.* **21** (1990) 823 – 836.
- [3] G.I. Barenblatt, Iu. P. Zheltov and I. N. Kochina, Basic concepts in the theory of seepage of homogeneous liquids in fissures rocks, *J. Appl. Math. Mech.* **24** (1960) 1286 – 1303.
- [4] M. A. Biot, Generalized theory of acoustic propagation in porous dissipative media, *Journal of Acoustic Society of America* **34** (1962) 1256 – 1264.
- [5] A. Bourgeat, L. Pankratov and M. Panfilov, Study of the double porosity model versus the fissures thikness, *Asymptotic Analysis* **38** (2004) 129 – 141.
- [6] R. Burridge and J. B. Keller, Poroelasticity equations derived from microstructure, *Journal of Acoustic Society of America* **70** No.4 (1981) 1140–1146.

- [7] Z. Chen Homogenization and simulation for compositional flow in naturally fractured reservoirs. *Math. Anal. App.* **326** (2007) 31 – 75.
- [8] H. Kazemi, Pressure transient analysis of naturally fractured reservoirs with uniform fracture distribution, *Soc. Petroleum Engrs. J.* **9**(1969) 451 – 462.
- [9] O.A. Ladyzhenskaya, *The mathematical Theory of Viscous Incompressible Flow*, (Gordon and Breach, New York, 1969).
- [10] A. Meirmanov, Nguetseng’s two-scale convergence method for filtration and seismic acoustic problems in elastic porous media, *Siberian Mathematical Journal* **48** (2007) 519 – 538.
- [11] A. Meirmanov, Acoustic and filtration properties of a thermoelastic porous medium: Biot’s equations of thermo – poroelasticity. *Sbornik Mathematics* **199**, No.3 (2008) 1 – 24.
- [12] A.Meirmanov, Homogenized models for filtration and for acoustic wave propagation in thermo-elastic porous media. *Euro. Jnl. of Applied Mathematics* **19** (2008) 259 – 284.
- [13] A.Meirmanov, A description of seismic acoustic wave propagation in porous media via homogenization. *SIAM J. Math. Anal.* **40** No.3(2008) 1272 – 1289.
- [14] G. Nguetseng, Asymptotic analysis for a stiff variational problem arising in mechanics. *SIAM J. Math. Anal.* **21**, (1990) 1394 – 1414.
- [15] E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, Vol. **129**, (Springer-Verlag, 1980).
- [16] A. de Swaan, Analytic solutions for determing naturally fractured reservoir properties by well testing. *Soc. Petroleum Engrs. J.* **16** (1976) 117 – 122.
- [17] J.E. Warren and P. J. Root, The behaviour of naturally fractured reservoirs. *Soc. Petroleum Engrs. J.* **3** (1963) 235 – 255.